Semester 3 Diff. eq.

The operator D : The use of symbol D for $\frac{d}{dx}$ is called operator.

The operators $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \dots$ are denoted by the Symbols $D^2, D^3, \dots$ Thus, $D^2y = \frac{d^2y}{dx^2}$, $D^3y = \frac{d^3y}{dx^3}$, ... and so on

Also, $\frac{1}{D}$ or D^{-1} or $\frac{1}{D^2}$ or D^{-2}

Note: $\frac{1}{D} \to$ Means Integration

$\frac{1}{D}(x) = \frac{x^2}{2}$.

Properties of D :-

(i) $D(u+v) = Du + Dv$ (distributive law)

(ii) $(D+a)u = Du + au = au + Du = (a+D)u$

(iii) $Dau = aDu$

(iv) $D^m D^n u = D^{m+n} u$

(v) $(D+\alpha)(D+\beta)u = (D+\alpha)(Du+\beta u)$

$= D^2u + D\beta u + \alpha Du + \alpha\beta u$

$= (D+\beta)(D+\alpha)u$

(v) But $Duv \neq uDv$.

Particular Integral (Right hand member a function of x)

– The given eqn be

$$D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + \dots + P_n y = X$$

OR, $f(D)y = X$

The expression $\frac{1}{f(D)}X$ will be used to denote a function of x not involving arbitrary constants

$\therefore$ The function $\frac{1}{f(D)}X$ is clearly a particular integral of the equation $f(D)y = X$.

Particular Case:- When $f(D) = D$, $\frac{1}{D}X$ will mean a function of x obtained by integrating X w.r.to x.

$\frac{1}{D}x^3 = \frac{x^4}{4}$, $\frac{1}{D^2}x^2 = \frac{1}{D}\cdot\frac{1}{D}(x^2) = \frac{1}{D}\cdot\left(\frac{x^3}{3}\right) = \frac{1}{3}\times\frac{x^4}{4}$, $\frac{1}{D}(1) = x$ etc.

Method of Find Particular Integrals (P.I.)

Let the diff. eq. $f(D)y = X$ —— ①

Then, $\frac{1}{f(D)} X$ is the Particular Integral (P.I.) of diff. eq ①

The P.I. depends on X and of following types:-

① $X = e^{ax}$

② $X = \sin ax$ or $\cos ax$

③ $X = x^m$, $m \in Z^+$

④ $X = e^{ax}.V$, V is any function of x

⑤ $X = x^m.V$; V is any function of x.

① To Find the P.I. : when $X = e^{ax}$, $a \in$ Constant.

we know that

$$D e^{ax} = a.e^{ax}$$

$$D^2 e^{ax} = D(De^{ax}) = D(ae^{ax}) = a^2 e^{ax}$$

$$= - - - - - - -$$

In general, $D^n e^{ax} = a^n.e^{ax}$. Then,

~~If f(D) is a polynomial~~

The P.I. of $f(D)y = e^{ax}$ is given by

$$y = \frac{1}{f(D)}.e^{ax} = \frac{1}{f(a)}.e^{ax}, \text{ provided } f(a) \neq 0$$

W.R. (i) Replace D by a

(ii) The Case when $f(a) = 0$ will be proceed in next type.

Ex ① Solve : $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x}$

Solution:- The given eqⁿ $D^2y + 3Dy + 2y = e^{2x}$ —— (I)

The auxiliary eqⁿ is

$$m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0$$

$$\Rightarrow m = -1, -2$$

∴ complementary function

$$C.F. = c_1 e^{-x} + c_2 e^{-2x} \quad \text{—— (II)}$$

Eq (1) becomes, $(D^2+3D+2)y = e^{2x}$

$\therefore$ The P.I. $y = \frac{1}{D^2+3D+2} e^{2x}$

$$= \frac{1}{2^2+3\times2+2} \cdot e^{2x} \quad \{\text{by Putting } D=2\}$$

$$= \frac{1}{12} \cdot e^{2x}$$

Hence, The general soln : $y = C.F. + P.I.$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{12} e^{2x} \quad \text{Ans}$$

Ex (2) Solve : $\frac{d^2y}{dx^2} + y = e^{-x}$

Soln:- The given eqn $\frac{d^2y}{dx^2} + y = e^{-x}$ —— (1)

The auxiliary eqn : $m^2+1=0 \Rightarrow m^2=-1 \Rightarrow m=\pm\sqrt{-1}=\pm i$

$$C.F. = A\cos x + B\sin x$$

For P.I. $(D^2+1)y = e^{-x}$

$$\Rightarrow y = \frac{1}{D^2+1} \cdot e^{-x}$$

$$\Rightarrow y = \frac{1}{(-1)^2+1} \cdot e^{-x} \quad \{\text{By Putting } D=-1\}$$

$$= \frac{1}{2} e^{-x}$$

$\therefore$ The general soln : $y = C.F. + P.I.$

$$y = A\cos x + B\sin x + \frac{1}{2} e^{-x}$$

Ans

Ex (3) Solve : $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{3x}$

soln:- Do yourself Ans: $y = (c_1+c_2x)e^{2x} + e^{3x}$

② To Find the P.I. : when $X = \sin ax$ or $\cos ax$.

$$D^2(\sin ax) = D\cdot(a\cos ax) = (-a^2)\sin ax$$

$$D^4(\sin ax) = (-a^2)^2\cdot \sin ax$$

In general. $(D^2)^n \sin ax = (-a^2)^n \sin ax$

Thus, if $\phi(D^2)$ is a polynomial in D^2, then $\phi(D^2)\sin ax = \phi(-a^2)\sin ax$, etc.

$\therefore$ P.I. of $f(D)y = \sin ax$ or $\cos ax$

$$y = \frac{1}{f(D)}\cdot \sin ax \text{ or } \frac{1}{f(D)}\cos ax$$

$$= \frac{1}{\phi(D^2)}\cdot \sin ax \text{ or } \frac{1}{\phi(D^2)}\cos ax$$

$$= \frac{1}{\phi(-a^2)}\cdot \sin ax \text{ or } \frac{1}{\phi(-a^2)}\cdot \cos ax$$

Note ① The Case when $f(D)$ Contains odd powers of D also.
(ii) If $\phi(-a^2) = 0$, we shall proceed in next Chapter.

Ex ① Solve: $\frac{d^2y}{dx^2} + y = \sin 2x$ —①

Soln:- The auxiliary eqn: $m^2 + 1 = 0$

$$\Rightarrow m = \pm i$$

$\therefore$ C.F. $= A\cos x + B\sin x$

From ① $(D^2+1)y = \sin 2x$

$$\therefore y = \frac{1}{(D^2+1)}\cdot \sin 2x$$

The P.I. $= \frac{1}{D^2+1}\cdot \sin 2x$

$= \frac{1}{-2^2+1}\cdot \sin 2x$ {Putting $D^2 = -a^2$ or $D^2 = -2^2$}

$= -\frac{1}{3}\sin 2x$

The general soln: $y = A\cos x + B\sin x - \frac{1}{3}\sin 2x$.

Ex ② Solve: $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$ —①

Soln:- The Auxiliary eqn.

$$m^2 + m + 1 = 0$$

$$\therefore m = \frac{-1 \pm \sqrt{1^2 - 4\times1\times1}}{2\times1} = \frac{-1\pm\sqrt{3}i}{2}$$

C.F. $= e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$= e^{-\frac{1}{2}x}\left\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}\right\}$ —②

Eq ① becomes, $(D^2+D+1)y = \sin 2x$

The P.I. $= \frac{1}{D^2+D+1}\cdot \sin 2x$

$= \frac{1}{-2^2+D+1}\cdot \sin 2x$ {Putting $D^2 = -2^2$}

$= \frac{1}{D-3}\cdot \sin 2x = \frac{D+3}{(D+3)(D-3)}\cdot \sin 2x$

$= \frac{D+3}{D^2-9}\cdot \sin 2x$

$= \frac{(D+3)}{-2^2-9}\cdot \sin 2x$

$= -\frac{1}{13}(2\cos 2x + 3\sin 2x)$ Ans